$$\frac{(1+\varepsilon\sqrt{p})^8+(1-\varepsilon\sqrt{p})^8}{(1-\varepsilon^2 p)^4},$$

where ε is a fourth root of unity, as only even powers of ε and $\sqrt{p}$ occur, and every even power of ε is $+1$ or -1. For $\varepsilon=+1$, or -1, we get the first value, and for $\varepsilon=+i$, or $-i$, we get the second value of $x+1/x$, and these are the only values which the formula admits. Thus

$$x+1/x=\frac{(1+\varepsilon\sqrt{p})^8+(1-\varepsilon\sqrt{p})^8}{(1-\varepsilon^2p)^4}=\frac{(1+\varepsilon\sqrt{p})^8+(1-\varepsilon\sqrt{p})^8}{(1-\varepsilon\sqrt{p})^4(1+\varepsilon\sqrt{p})^2}$$

$$=\left(\frac{1+\varepsilon\sqrt{p}}{1-\varepsilon\sqrt{p}}\right)^4+\left(\frac{1+\varepsilon\sqrt{p}}{1+\varepsilon\sqrt{p}}\right)^4=\lambda+1/\lambda$$

if λ denote $\left(\frac{1+\varepsilon\sqrt{p}}{1-\varepsilon\sqrt{p}}\right)^4$. Then as before

$$x^2-[\lambda+(1/\lambda)]x+1=(x-\lambda)[x-(1/\lambda)]=0,$$

and $x=\lambda$, or $x=1/\lambda$, but when ε takes its four values, the group of values represented by λ, is the same as the group represented by $1/\lambda$, though not corresponding value by value for the same value of ε. The values of λ and $1/\lambda$ are the same for 1, and -1, -1 and 1, i and $-i$, $-i$ and i, respectively.

The six values of x are

$$x=p^2,\ x=1/p^2,\ \text{and either } x=\left(\frac{1+\varepsilon\sqrt{p}}{1-\varepsilon\sqrt{p}}\right)^4,\ \text{or } x=\left(\frac{1-\varepsilon\sqrt{p}}{1+\varepsilon\sqrt{p}}\right)^4$$

for the last four values, or other similar expressions containing ε, which are easily formed.

GEOMETRY.

119. Proposed by WILLIAM HOOVER., A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

A sphere touches each of two straight lines which are inclined to each other at a right angle but do not meet; show that the locus of its center is an hyperbolic paraboloid.

I. Solution by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

If the common perpendicular to the two given lines be taken as the Z-axis, its length represented by $2c$, its middle point be taken as origin, a plane parallel to the two lines as xy-plane, and the projections of the lines on the xy-plane as the X and Y-axis, respectively, the equations of the lines will be $y=0$, $z=c$, and $x=0$, $z=-c$. The square of the distance of the point (x', y', z'), any point in the required locus, from the first line is $y'^2+(c-z')^2$, and from the second

line $x'^2+(c+z')^2$. These distances are equal, and we get $x^2-y^2+4cz=0$ as the required equation. This is the equation of a hyperbolic paraboloid.

A more general problem is to find the locus of points equi-distant from any two non-intersecting straight lines in space.

If the axes are taken as above, except that the bisectors of the angles formed by the projections of the given lines on the xy-plane are taken as the X and Y-axis, the resulting equation will be

$$\frac{mxy}{1+m^2}+cz=0,$$

which is also the equation of a hyperbolic paraboloid.

II. **Solution by the PROPOSER.**

With coördinate axes rectangular, the two given lines may be taken as

$$y=mx,\ z=c\ldots\ldots(1),\ \text{and}\ y=-mx,\ z=-c\ldots\ldots(2).$$

Let the sphere be $(x-x')^2+(y-y')^2+(z-z')^2=r^2\ldots\ldots(3)$.

(1) intersects (3) where

$$(1+m^2)x^2-2(x'+my')x+(x'^2+y'^2+c^2-2cz'+z'^2-r^2)=0\ldots\ldots(4).$$

(1) will then be tangent to (3) if

$$(x'+my')^2=(1+m^2)(x'^2+y'^2+c^2-2cz'+z'^2-r^2)\ldots\ldots(5).$$

Similarly, (2) will be tangent to (3) if

$$(x'-my')^2=(1+m^2)(x'^2+y'^2+c^2+2cz'+z'^2-r^2)\ldots\ldots(6).$$

(5)−(6) gives, $mx'y'=-c(1+m^2)z'\ldots\ldots(7)$, the required locus of the center (x', y', z') of (3). But (7) is an hyperbolic paraboloid.

III. **Solution by J. W. YOUNG, Graduate Student, Ohio State University, Columbus, Ohio.**

Let the two straight lines be

$$\frac{x}{0}=\frac{y}{0}=\frac{z}{1};\quad \frac{x-a}{0}=\frac{y}{1}=\frac{z}{0}.$$

The center of a sphere, which touches these two straight lines will always be equidistant from them. Hence equating distances, putting (x_1, y_1, z_1) for the center, we have

$$x_1^2+y_1^2=(x_1-a)^2+z_1^2.$$

Whence the required locus is easily seen to be

$$y^2-z^2=-2ax+a^2,$$

a hyperbolic paraboloid.

Excellent solutions were received from *G. B. M. ZERR*, and *J. SCHEFFER*.